Sato-Tate Groups and Distributions of $y^\ell = x(x^\ell - 1)$

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Objective

Define $C_\ell/\mathbb{Q}$ as the curve $y^\ell = x(x^\ell - 1)$, where ℓ is an odd prime.

Our goal is to study the limiting distributions of the coefficients of the normalized L-polynomials associated with C_ℓ. Motivated by the findings of this objective, we provide further results on the number of points on C_ℓ over various $\mathbb{F}_q$, where q is an odd prime.

Background

L-Polynomials

For primes p of good reduction for C, the zeta function of C is

$$Z(C/\mathbb{F}_p, T) := \exp\left(\sum_{k=1}^{\infty} \frac{\#C(\mathbb{F}_{p^k})T^k}{k}\right) = \frac{\mathbf{L_p(C,T)}}{(1-T)(1-pT)}.$$

Define the normalized L-polynomial as

$$\begin{aligned}\overline{L}_p(C,T) &:= \mathbf{L_p(C, T/\sqrt{p})}\\ &= T^{2g} + a_1 T^{2g-1} + a_2 T^{2g-2} + \cdots + a_2 T^2 + a_1 T + 1,\end{aligned}$$

where $a_i \in \left[-\binom{2g}{i}, \binom{2g}{i}\right]$ and g denotes the genus of C.

The coefficients of $\overline{L}_p(C,T)$ contain important arithmetic information about C

- The a_1 coefficient is the *trace of Frobenius*:

$$a_1 = p + 1 - \#C(\mathbb{F}_p).$$

Goal: Find the limiting distribution of a_i as $p \to \infty$.

Moment Statistics

We can describe the distributions numerically via *moment statistics*.

The nth moment (about 0) of an independent random variable X is $M_n[X] = \mathbb{E}[X^n]$. For a dataset $X = \{x_i\}_{i=1}^m$, the nth moment can also be expressed as

$$M_n[X] = \frac{1}{m}\sum_{i=1}^{m} x_i^n,$$

and for any constant c, $M_n[c] = c^n$.

The Sato-Tate Conjecture

The Sato-Tate conjecture is a statistical statement about the distribution of the number of points on the reduction modulo primes of an elliptic curve defined over $\mathbb{Q}$.

The *generalized* Sato-Tate conjecture attempts to extend this statement to any abelian variety. This version claims the existence of a compact Lie group, the *Sato-Tate group*, that can determine the limiting distribution of the normalized L-polynomials' coefficients via the *Haar measure*.

Our curve, C_ℓ, and its Jacobian are equipped with

- complex multiplication (CM)
- nondegeneracy.

The conjecture is proven to be true for abelian varieties with CM. In addition, there are well-established techniques for computing Sato-Tate groups for nondegenerate abelian varieties.

Preliminaries

Our curve C_ℓ isn't an abelian variety. But, the *Jacobian*, $\mathrm{Jac}(C_\ell)$, is.

- $\mathrm{Jac}(C_\ell)$ is nondegenerate.

A curve automorphism of C_ℓ is

$$\alpha : (x, y) \mapsto (\zeta^\ell x, \zeta^{\ell+1} y),$$

where $\zeta := \zeta_{\ell^2}$.

We define a space of regular differential 1-forms, $\mathcal{B}$, as

$$\mathcal{B} = \left\{\omega_{a,b} := x^a \frac{dx}{y^b} \;\middle|\; \begin{array}{c} 0 \le a \le \ell - 2, \\ a + 1 \le b \le \ell - 1 \end{array}\right\} \implies \alpha^*\omega_{a,b} = \zeta^{\ell(a+1-b)-b}\omega_{a,b}.$$

An endomorphism, α, of $\mathrm{Jac}(C_\ell)$ is then

$$\alpha[j,j] = Z^{\ell(a_j+1-b_j)-b_j},$$

where $0 \le j \le g-1$, $e_j := \langle \ell(a_j + 1 - b_j) - b_j\rangle_{\ell^2}$, and $S := \{e_j\}$.

For $\langle \sigma_n \rangle = \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^2})/\mathbb{Q})$, define $g_i := \langle \ell n(a_i + 1 - b_i) - nb_i\rangle_{\ell^2} = \langle ne_i\rangle_{\ell^2}$. Then, the Galois action on the endomorphism α is

$$^{\sigma_n}\alpha[i,i] = \begin{cases} Z^{g_i} & \text{if } g_i \in S \\ \overline{Z}^{\ell^2 - g_i} & \text{if } g_i \notin S. \end{cases}$$

Since $\mathrm{Jac}(C_\ell)$ is nondegenerate, we can compute the *twisted Lefschetz group* to get $\mathrm{ST}(\mathrm{Jac}(C_\ell))$.

Theorem - The Sato-Tate Group [1]

The **Sato-Tate group**, up to isomorphism in $\mathrm{USp}(2g)$, of $\mathrm{Jac}(C_\ell)$ is

$$\mathrm{ST}(\mathrm{Jac}(C_\ell)) \cong \langle \mathrm{U}(1)^g, \gamma\rangle,$$

where γ is a $2g \times 2g$ block-signed permutation matrix whose ij^{th} block is

$$\gamma[i,j] = \begin{cases} I & \text{if } g_i = e_j \\ J & \text{if } g_i = \ell^2 - e_j \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \le i, j \le g-1$.

Distributions

Let $U \in U(1)^g$ where $U = \mathrm{diag}(u_0, \overline{u}_0, u_1, \overline{u}_1, \ldots, u_{g-1}, \overline{u}_{g-1})$, and denote $\alpha_i := u_i + \overline{u}_i$.

We find the limiting distributions of the L-polynomials by the charpolys of the elements of $\mathrm{ST}(\mathrm{Jac}(C_\ell))$: $U \cdot \gamma^i$, where $0 \le i < \ell(\ell-1)$.

The a_1 Characteristc Polynomial Coefficient

For $i > 0$, $U \cdot \gamma^i = 0$, so $M_n[{}^i\mu_1] = 0$. For $i = 0$, the a_1 coefficient is $\mathrm{tr}(U) = -(\alpha_0 + \alpha_1 + \ldots + \alpha_{g-1})$.

The moment sequence for each α_i is that of the unitary group $\mathrm{U}(1)$:

$$M[\mu_{\mathrm{U}(1)}] = [1, 0, 2, 0, 6, 0, 20, \ldots].$$

By properties of the expected value, we get

$$M_n[\mu_1] = \frac{1}{\ell(\ell-1)} \sum_{b_0+b_1+\ldots+b_{g-1}=n} \binom{n}{b_0, b_1, \ldots, b_{g-1}} M_{b_0}[\alpha_0] M_{b_1}[\alpha_1] \ldots M_{b_{g-1}}[\alpha_{g-1}].$$

Example: $\ell = 5$

The genus of C_5 is $g = \phi(5^2)/2 = 5(4)/2 = 10$, meaning

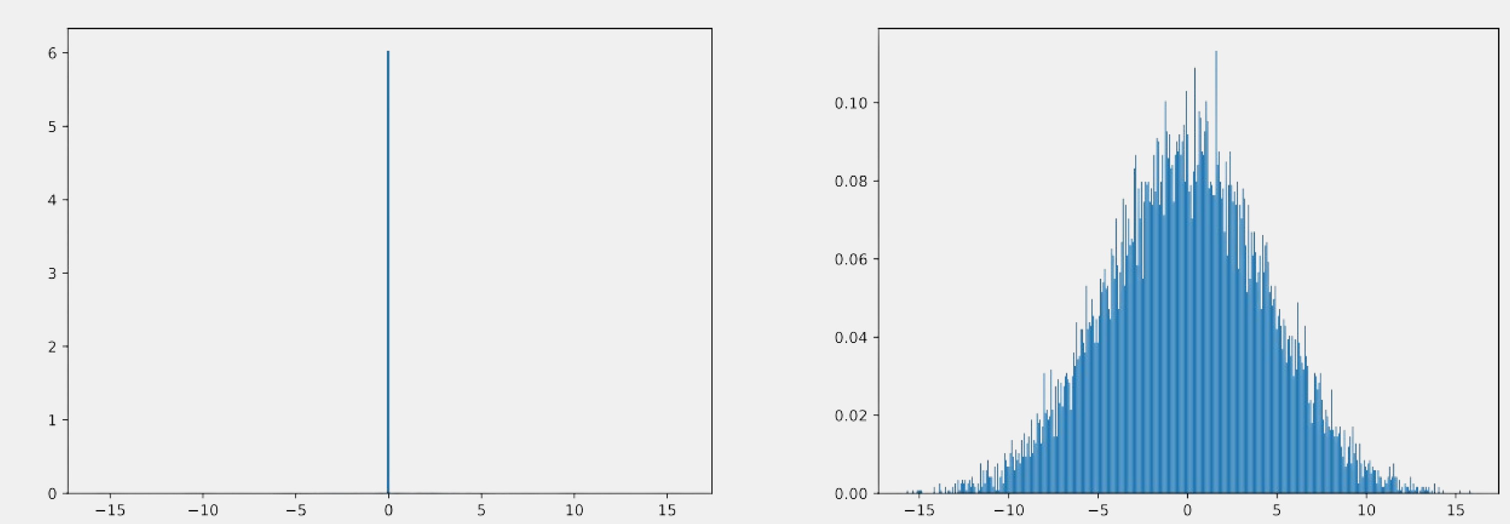
$$\mathrm{ST}(\mathrm{Jac}(C_5)) \cong \langle \mathrm{U}(1)^{10}, \gamma\rangle.$$

The μ_1 moment statistics generated by $\mathrm{ST}(\mathrm{Jac}(C_5))$ are

$$[1, 0, 1, 0, 57, 0, 5140, 0, 615545, \ldots].$$

	M_2	M_4	M_6	M_8
a_1	1.00479	58.4085	5363.22	646563
μ_1	1	57	5140	615545

Table 1. The a_1 and μ_1 moments ($p < 2^{22}$).



(a) For all primes q.

(b) For primes $q \equiv 1 \pmod{25}$.

Figure 1. Histograms of the a_1 coefficients over different $\mathbb{F}_q$.

Theorem - Point Count [1]

If $q \not\equiv 1 \pmod{\ell^2}$, then $\#C_\ell(\mathbb{F}_q) = q + 1$.

Next Steps

- Compute moment statistics for the $a_2, a_3, \ldots, a_g$ coefficients when $\ell = 5, 7, \ldots$
- Improve bounds for computing the numerical moments.
- Study other families of curves!

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References

[1] Heidi Goodson and Rezwan Hoque. Sato-tate groups and distributions of $y^\ell = x(x^\ell - 1)$. *arXiv e-prints*, page arXiv:2412.02522, 2024.